SOLUTION

TQ1 [5 points]

1.1 Based on Einstein's relationship, E=mc^2. The relationship is Doing a mass-energy balance:

$$35M_{\odot}C^2 + 30M_{\odot}C^2 = 62M_{\odot}C^2 + E_{GW}$$ [1 point]

$$3M_{\odot}C^2 = E_{GW}$$ [1 point]

$$M_{\odot} = 1.988\times10^{30}\ kg$$

$$E_{GW} = 3(1.988\times10^{30}\ kg)(9\times10^{16}\ \tfrac{m^2}{s^2})$$ [1 point]

$$E_{GW} = 5.36\times10^{47} J$$ [1 point]

Now using $E_{SN} = 2\times10^{44} J$ we have the ratio:

$$\frac{E_{SN}}{E_{GW}} = 3.7\times10^{-4}$$ [1 point]

GW150914 released approximately 2865 times more energy than a supernova explosion. $E_{SN} << E_{GW}$.

SOLUTION

TQ2 [10 points]

2.1 The solar radiation intensity (I_T) received on Earth is:

$$I_T = \frac{P_s}{4\pi r_{s-t}^2} = \sigma T_s^4 \bullet \left(\frac{R_s}{r_{s-t}}\right)^2 \qquad \text{[1 point]}$$

with $P_s = 4\pi\sigma R_s^2 T_s^4$.

Also, Earth would absorb energy at this rate:

$$P_{abs} = I_T \pi R_t^2 = \pi\sigma T_s^4 \bullet \left(\frac{R_s \bullet R_t}{r_{s-t}}\right)^2$$

With $R_t = 6.4\times10^6 m$ as the radius of the planet "disk". [1 point]

Then, by thermal equilibrium, the absorbed radiation would be radiated over the planet's surface:

$$P_{abs} = P_{rad}$$

With

$$P_{rad} = 4\pi\sigma R_t^2 T_t^4$$

$$\pi\sigma T_s^4 \bullet \left(\frac{R_s \bullet R_t}{r_{s-t}}\right)^2 = 4\pi\sigma R_t^2 T_t^4 \qquad \text{[1 point]}$$

$$T_t = T_s \cdot \left(\frac{R_s}{2\, r_{s-t}}\right)^{\frac{1}{2}} = 278.58\ K = 5.43\ °C \qquad \text{[1 point]}$$

This would be very cold but still viable to harbor life.

2.2 The Earth's absorbed radiation, considering the albedo, is:

$$P'_{abs} = 0.7 \cdot \pi\sigma T_s^4 \bullet \left(\frac{R_s \bullet R_t}{r_{s-t}}\right)^2 \qquad \text{[1 point]}$$

$$T'_T = T_s\,(0.7)^{\frac{1}{4}} \bullet \left(\frac{R_s}{2\, r_{s-t}}\right)^{\frac{1}{2}} = 254.81\ K = -18.34\ °C \qquad \text{[1 point]}$$

2.3 If Earth reabsorbs 58% of the 70% re emitted energy, then:

$$T''_T = T_s\,[0.7 + (0.58 \cdot 0.7)]^{\frac{1}{4}} \bullet \left(\frac{R_s}{2\, r_{s-t}}\right)^{\frac{1}{2}} = 285.68\ K = 12.53\ °C \qquad \text{[4 points]}$$

SOLUTION

TQ3 [10 points]

3.1 Since the path between A and B is parabolic, the total energy of the spacecraft is zero,

$$E_{AB} = 0, \qquad \leftrightarrow \quad \varepsilon = 1.$$

So, when you get to point B, then

$$E = \tfrac{1}{2}mv_B^2 - \tfrac{GMm}{r_B} = 0; \qquad \text{[1 point]}$$

From where

$$v_B = \sqrt{\tfrac{2GM}{r_B}} \qquad \text{[1 point]}$$

Replacing with available values

$$v_B = \sqrt{\frac{2 \times (6{,}67\times10^{-11}) \times (6{,}4\times10^{23})}{6{,}8\times10^{6}}} \approx 3,55\ km\ s^{-1} \qquad \text{[1 point]}$$

3.2 There are two possible solutions

Solution A

Immediately after breaking the total energy and the angular momentum change since the braking force is tangential, but along the elliptical path BC the values with which it passes through are conserved and take on during the entire journey. C; then applying conservation of energy between points B' (immediately after braking) and C as well as conservation of the angular momentum between B' and C, we have

$$\tfrac{1}{2}mv_{B'}^2 - \tfrac{GMm}{r_B} = \tfrac{1}{2}mv_C^2 - \tfrac{GMm}{R_M} \quad Eq.\,1 \qquad \text{[1 point]}$$

$$L = mv_{B'}r_B = mv_C R_M \quad Eq.\,2 \qquad \text{[1 point]}$$

From these two equations we solve for v_C and it's result is used to calculate the total energy in C which gives us:

$$E_{elipse} =- \tfrac{GMm}{R_M + r_B} \approx \ -2.10 \times 10^{11} J \quad Eq.\,3a \qquad \text{[2 points]}$$

In both methods, this and next, two points for getting the equation and two points for calculation + negative sign + unit.

SOLUTION

TQ4 **[10 points]**

4. 1 To get the number of photons per second we have to multiply the Flux by the area of the dish, to know the incoming energy per second, and divide this by the energy of a photon.

For $\lambda_1 = 0.32\ mm = 3.2\times10^{-4} m$:

Energy of a photon: $E = \frac{hc}{\lambda} = 6.2076 \times 10^{-22} Jules$ [1 point]

Area of the disk: $A = \pi R^2 = 113\ m^2$

Number of photons per second:

$$n_1 = \frac{flux\times A}{E} \approx 1820\ photon/s$$ [1 point]

4.2 Same calculation, just changing the energy of the individual photons:

For $\lambda_2 = 8.6\ mm = 8.6 \times 10^{-3} m$

Energy of a photon: $E = \frac{hc}{\lambda} = 2.31 \times 10^{-23} Jules$ [1 point]

Number of photons per second:

$$n_2 = \frac{flux\times A}{E} \approx 48900\ photon/s$$ [1 point]

4.3 Spatial resolution of a single telescope is given by:

$$\theta = 1.22\frac{\lambda}{D}$$

where D represents the diameter of the dish. This value will be given in radians, so it must be converted to arcsec afterwards.

For a frequency of 74.9 GHz, the corresponding wavelength is:

$$\lambda = \frac{c}{f} = 4\ mm$$ [1 point]

And the spatial resolution:

$$\theta = 1.22\frac{4mm}{12\ m} \approx 83.9\ arcsec$$ [1 point]

4.4 For an array the correct expression is:

$$\theta = \frac{\lambda}{B} \quad \text{or} \quad \theta = \frac{\lambda}{2B}$$ [1 point]

being B the longest baseline in the array.

So in this case:

$$\theta = \frac{4\,mm}{16\,km} \approx 0.0516\ arcsec \quad \text{or} \quad 0.0258\ arcsec$$ [1 point]

4.5 The SEFD is a characteristic flux of a system, found by dividing the characteristic energy associated to the so-called temperature of the antenna by its effective area:

$$SEFD = \frac{2kT_{sys}}{A_e}$$ [1 point]

As no additional information is given about the effective area, the actual physical area of the array should be used:

$$A = 54\left(\pi\times 6^2\right) + 12\left(\pi\times 3.5^2\right) \approx 6569\ m^2$$ [0.5 point]

Substituting the Boltzmann constant, the given temperatura of the antenna, and converting the answer to Jansky we get:

$$SEFD \approx 290.5\ Jy$$ [0.5 point]

NOTES:

Question	We indicate the answer must be:	Tolerance
4.a	Approximated to the nearest integer	+/- 10 photons
4.b	Approximated to the nearest integer	+/- 10 photons
4.c	2 digit of precision	[83.0 , 85.0]
4.d	2 digits of precision	0.05 exact
4.e	2 digit of precision	[289.0 , 291.0]

SOLUTION

TQ5 [10 points]

5.1 Pressure inside the flux tube = Pressure outside the flux tube (surroundings)

(Magnetic pressure + Gas pressure inside) = Gas pressure outside

Particularly,

$$\frac{B_0^2}{2\mu_0} + P_{gas_{in}} = P_{gas_{out}}$$ [2 points]

$$\frac{B_0^2}{2\mu_0} = P_{0_{out}} - P_{0_{in}}$$ Eq. (1)

Similarly, for a particular height z,

$$\frac{B(z)^2}{2\mu_0} = e^{-z/H}\left(P_{0_{out}} - P_{0_{in}}\right)$$ Eq. (2)

[2 points]

Dividing equation (2) to (1),

$$\frac{B(z)^2}{B_0^2} = e^{-z/H}$$

To finally get

$$B(z) = B_0 e^{-z/2H}$$ [3 points]

5.2 At $z = H$, $B = B_0 e^{-z/150}$

$$0.03 = 0.3\, e^{-\frac{z}{2x150}}$$

$$z = 300 \times ln\ 10$$

$$z = 300 \times 2.301$$

$$z = 690\ km$$ [3 points]

SOLUTION

TQ6 [12 points]

6.1 From Wien's law:

$$\lambda_{max} = \frac{2.898\times10^{-3} m\cdot K}{T_{eff}(K)} = \frac{2.898\times10^{-3} m\cdot K}{4995\ K}$$

$$\lambda_{max} = 5.8\times10^{-7} m = 580.2\ nm$$ [2 point]

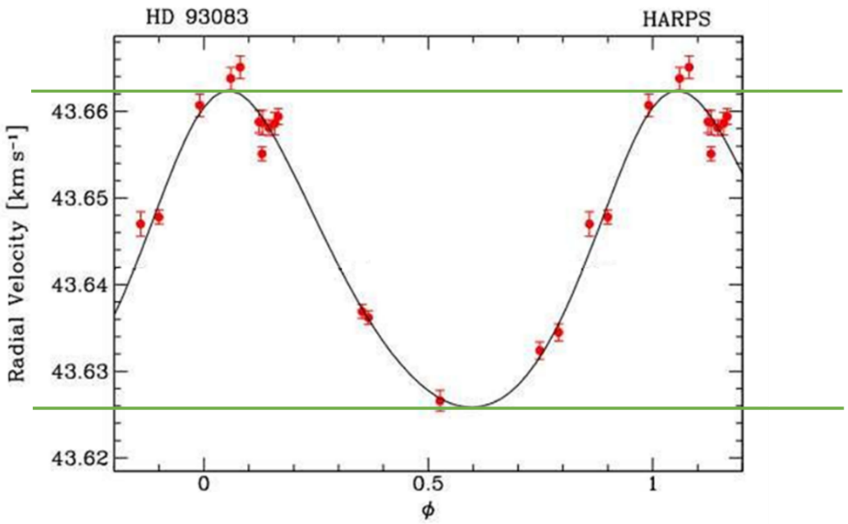
6.2 With the parallax doable to find the distance from Earth:

$$d = \frac{1}{pllx\ (arcsec)} = \frac{1}{0.035\ arcsec} = 28.55\ pc$$ [1 point]

$$M_V = m_v - 5\ log(d[pc]) + 5$$

$$M_V = 8.3 - 5\ log(28.57) + 5 = 6.02$$ [1 point]

6.3 [2 point]



Each division on Y has 0.002 kms^{-1}, so the maximum radial velocity is 43.662 kms^{-1} and the minimus is 43.626 kms^{-1}. Then, the mean radial velocity of Macondo is $v_r = 43.644\ km\ s^{-1}$.

6.4 The tangential velocity of Macondo, from the plot, is its variation from the mean system velocity

$$v_s = 43.662\ km\ s^{-1} - 43.644\ km\ s^{-1} = 0.018\ km\ s^{-1}$$

The masses of the star and the planet are known so it is only needed to find orbital velocity of Melquiades. But first it is important to convert the mass of the star to kg.

$$0.7M_\odot \times \frac{1.989\times10^{30}kg}{1M_\odot} = 1.4 \times 10^{30}kg$$ [1 point]

$$v_p = \frac{m_s}{m_p} \times v_s = \frac{1.4\times10^{30}\ kg}{7\times10^{26}\ kg} \times 0.02\ km\cdot s^{-1} = 40\ km\cdot s^{-1}$$ [1 point]

6.5 As the motion of the planet is circular, the orbital period is:

$$T = \frac{2\pi}{v_p} \times a$$

being ***a*** the distance to the central star.

With the 3rd Kepler's law, the orbital period is $T^2 = \frac{4\pi^2}{Gm_s} \times a^3$

Then, combining both expressions of *T* is possible to find *a*

$$\left(\frac{2\pi}{v_p} \times a\right)^2 = \frac{4\pi^2}{Gm_s} \times a^3$$

$$\left(\frac{2\pi}{v_p}\right)^2 \frac{Gm_s}{4\pi^2} = a$$

$$a = 7.25\times10^{10}\ m = 0.48\ au$$ [2 point]

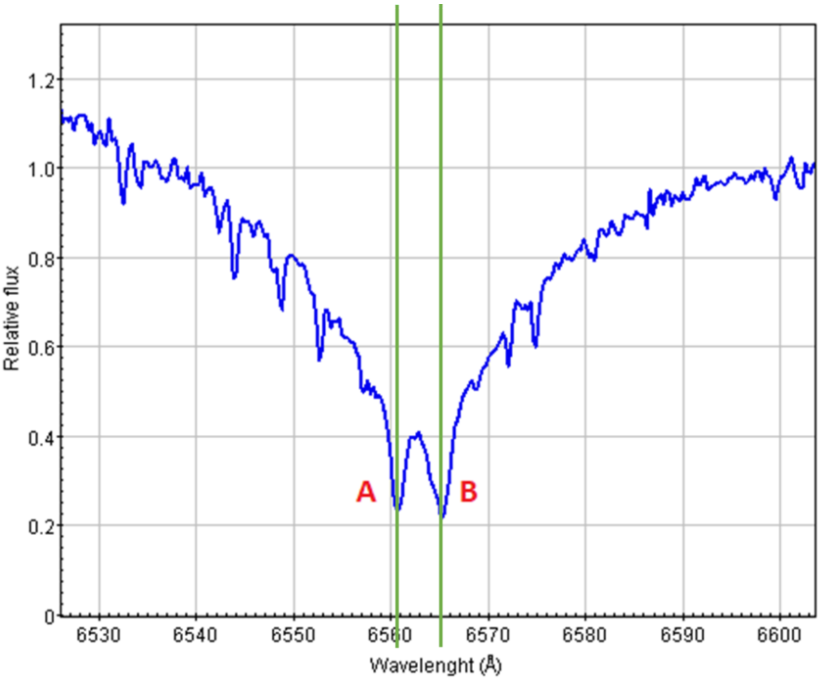
With the previous solution is able to find the orbital period:

$$T = \frac{2\pi}{v_p} \times a = \frac{2\pi\times(7.21x10^{10}m)}{36000\ m\cdot s^{-1}} = 1.27 \times 10^7 s = 147\ days$$ [2 point]

SOLUTION

TQ7 **[13 point]**

7.1 Observing the plot, it is possible to estimate the wavelength of both Hα lines and later find the velocity of each star by Doppler.



$$v_A = \frac{6560.4\ \text{Å} - 6562.8\ \text{Å}}{6562.8\ \text{Å}} \times (3 \cdot 10^5\ km \cdot s^{-1}) = -110\ km \cdot s^{-1}$$ [2 point]

$$v_B = \frac{6565.4\ \text{Å} - 6562.8\ \text{Å}}{6562.8\ \text{Å}} \times (3 \cdot 10^5\ km \cdot s^{-1}) = +119\ km \cdot s^{-1}$$ [2 point]

So, Menkalinan A is approaching and Menkalinan B is moving away from us.

[1 point]

7.2 Applying the 3rd Kepler's law, the total mass of the system is :

$$m_A > m_B \Rightarrow \alpha_B > \alpha_A$$

$$\frac{\alpha_B}{\alpha_A} = 1.026$$

$$\alpha = (\alpha_A + \alpha_B)/2 = \frac{\alpha_B}{2}(1 + \frac{\alpha_A}{\alpha_B}) = 1.97\frac{\alpha_B}{2}$$ [2 point]

$$M_{tot} = \frac{4\pi^2}{G} \times \frac{\left(0.00330'' \times \frac{1rad}{206265''}\right)^3 \bullet \left(81.1ly \cdot \frac{9.46 \bullet 10^{15} m}{1ly}\right)^3}{\left(3.96d \bullet \frac{86400s}{1d}\right)^2}$$

$M_{tot} = 9.41 \cdot 10^{30} kg = 4.73 M_\odot$ [2 point]

7.3

$$m_1 / m_2 = 1.026$$

$$M_{tot} = m_1 + m_2 = 1.026 \cdot m_2 + m_2 = 2.026\, m_2 = 9.41 \cdot 10^{30}\ kg$$

$m_2 = 1.014 \cdot 10^{31}\ kg / 2.026 = 4.64 \times 10^{30}\ kg = 2.34 M_\odot$ [1 point]

$m_1 = 1.026 \cdot m_2 = 2.40\ M_\odot$ [1 point]

7.4 Using the mass/luminosity relation given:

$$\frac{L_1}{L_\odot} = (\frac{m_1}{M_\odot})^{3.5} = (\frac{2.40 M_\odot}{M_\odot})^{3.5} = 21.3$$

$L_1 = 21.3\, L_\odot$ [1 point]

$$\frac{L_2}{L_\odot} = (\frac{m_2}{M_\odot})^{3.5} = (\frac{2.34 M_\odot}{M_\odot})^{3.5} = 19.5$$

$L_2 = 19.5 L_\odot$ [1 point]

SOLUTION

TQ8 [15 points]

8.1 Assuming R=6378 km (as per the table of constants)

$$\Delta x = R \cdot [(4°36'18'') - (4°35'30'')] \cdot \frac{\pi}{180} = 1.4842\ km$$

$$\Delta y = R \cdot [(74°3'19'') - (74°3'15'')] \cdot \frac{\pi \cdot cos(4,35,54)}{180} = 0.1237\ km$$

$$\Delta z = (3.100 - 3.296) = -0.196\ km$$

[2 points]

$$d_{2-3} = \sqrt{(x^2 + y^2 + z^2)} = 1.502\ km$$ [1 point]

8.2 Estimate the angular separation (in degrees) between Guadalupe (2) and Monserrate (3) as observed from the National Astronomical Observatory of Colombia (1). Take point 1 as the vertex.

Using the same method as part 8.1:

$$d_{1-2} = 2.722\ km$$ [2 points]

$$d_{1-3} = 2.580\ km$$ [2 points]

Using Cosine rule for spherical triangle,

$$cos(A) = \frac{({d_{1-2}}^2 + {d_{1-3}}^2 - {d_{2-3}}^2)}{2\ d_{1-2}\ d_{1-2}} = 0.841$$

$$A = 32.78°$$ [2 points]

8.3

$$\alpha = tan^{-1}\left(\frac{-sin(\beta)\cdot sin(\varepsilon) + cos(\beta)\cdot cos(\varepsilon)\cdot sin(\lambda)}{cos(\beta)\cdot cos(\lambda)}\right)$$

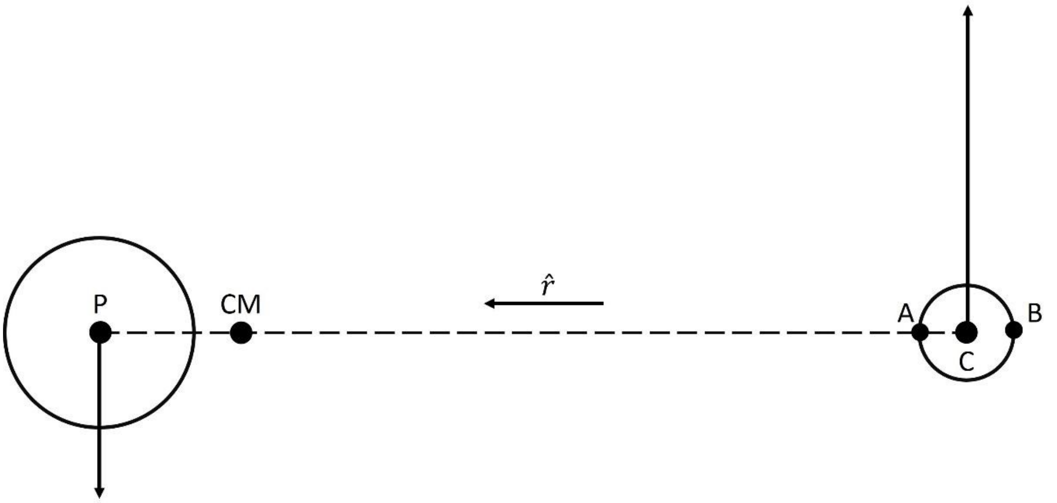
$$\alpha = 12°43'3'' = 0h\ 50m\ 52s$$ [3 points]

$$\delta = sin^{-1}(sin(\beta) \cdot cos(\varepsilon) + cos(\beta) \cdot sin(\varepsilon) \cdot sin(\lambda))$$

$$\delta = 1°33'43''$$ [3 points]

SOLUTION

TQ9 [15 points]



Let P and C be the centers of Pluto and Charon respectively, and CM their center of mass. Clearly $\overline{CM - P} = \frac{1}{9}R$ and $\overline{CM - C} = \frac{8}{9}R$. The effective gravitational field at points A and B are given by:

[By recognizing the vectorial nature of gravities/forces/fields] [4 points]

$$g_A = \frac{GM_P}{(R-r)^2}\hat{r} - \frac{GM_C}{r^2}\hat{r} - \omega^2\left(\tfrac{8}{9}R - r\right)\hat{r} \quad \text{[1 point]}$$

$$g_B = \frac{GM_P}{(R+r)^2}\hat{r} + \frac{GM_C}{r^2}\hat{r} - \omega^2\left(\tfrac{8}{9}R + r\right)\hat{r} \quad \text{[1 point]}$$

where $\omega^2 = 9\frac{GM_C}{R^3}$ is the angular speed common to both Pluto and Charon. Thus, the gravitational accelerations at these points are:

[by finding angular speed] [2 points]

$$g_A = \frac{GM_P}{(R-r)^2} - \frac{GM_C}{r^2} - \frac{GM_C}{R^2}\left(8 - 9\tfrac{r}{R}\right)$$

$$g_B = \frac{GM_P}{(R+r)^2} + \frac{GM_C}{r^2} - \frac{GM_C}{R^2}\left(8 + 9\tfrac{r}{R}\right)$$

Factorizing the expression to obtain formulas proportional to Charon's mass [1 point]:

$$g_A = -\frac{GM_C}{r^2}\left[1 - \frac{M_P}{M_C}\left(\frac{r}{R}\right)^2 \frac{1}{\left(1-\frac{r}{R}\right)^2} + \left(\frac{r}{R}\right)^2\left(8 - 9\frac{r}{R}\right)\right]$$

$$g_B = \frac{GM_C}{r^2}\left[1 + \frac{M_P}{M_C}\left(\frac{r}{R}\right)^2 \frac{1}{\left(1+\frac{r}{R}\right)^2} - \left(\frac{r}{R}\right)^2\left(8 + 9\frac{r}{R}\right)\right]$$

Replacing the numerical value of the ratio of the two masses:

$$g_A = -\frac{GM_C}{r^2}\left[1 - \left(\frac{r}{R}\right)^2 \frac{8}{\left(1-\frac{r}{R}\right)^2} + \left(\frac{r}{R}\right)^2\left(8 - 9\frac{r}{R}\right)\right]$$

$$g_B = \frac{GM_C}{r^2}\left[1 + \left(\frac{r}{R}\right)^2 \frac{8}{\left(1+\frac{r}{R}\right)^2} - \left(\frac{r}{R}\right)^2\left(8 + 9\frac{r}{R}\right)\right]$$

Noting that the expressions in square brackets are dimensionless, it is helpful to replace the given values of r and R, resulting [2 points by expressions of g_A and g_B]:

$$g_A = -\frac{GM_C}{r^2}\,(0,99929)$$

$$g_B = \frac{GM_C}{r^2}\,(0,99933)$$

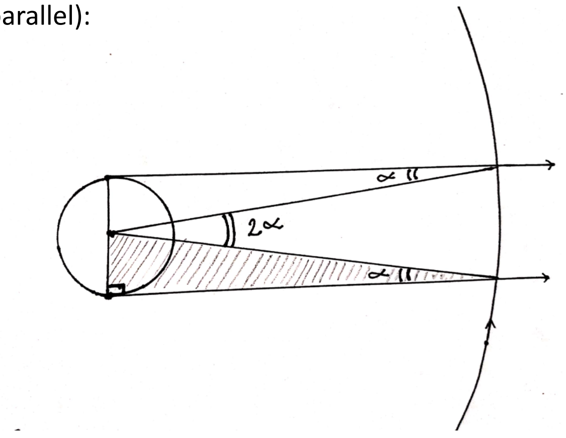
The percentage difference with respect to Charon's normal gravity $g_0 = \frac{GM_C}{r^2}$ is

$$\frac{\left||g_A| - |g_B|\right|}{g_0}\cdot 100\% = 0,004\% = 4\times 10^{-3}\% \qquad \text{[1 point]}$$

SOLUTION

TQ 10 [15 points]

10.1 The maximum duration occurs if Earth passes exactly along the diameter of the Sun. Now consider the following figure, where Sun rays are depicted as traveling to the right towards the hypothetical distant observer (reason why they can be considered parallel):



The arc along the circumference associated to the transit, 2α, can be easily found looking at the shaded triangle:

$$\alpha = sin^{-1}(\frac{R_\odot}{R_{orb}})$$ [2 points]

being R_{orb} the orbital radius of the Earth. Now the time of the transit can be found by considering the angular velocity of the Earth, for instance by means of the following proportionality relations:

$$\frac{t_{tr}}{T_{orb}} = \frac{2\alpha}{2\pi} = \frac{sin^{-1}(\frac{R_\odot}{R_{orb}})}{\pi}$$ [2 points]

Substituting with the known values for these quantities we get, i.e.,

$$t_{tr} \sim 12.97\ h = 12h\ 58m$$ [1 point]

10.2 First of all it must be said that the minimum orbital period is obtained if we assume that the transit occurs along the diameter of the star. In other cases, the planet would be crossing a shorter path in front of the star during the same time, meaning that it would have a smaller angular velocity and therefore a longer period.

That said, it means that we can resort to the same expression of 10.1, yet this time we need 2 additional elements:

- Using the small angle approximation:

$$sin(\alpha) \sim \alpha$$ [1 point]

- Invoking the full expression for the orbital period:

$$T_{orb} = \frac{2\pi}{\sqrt{GM_*}} R_{orb}^{3/2}$$ [1 point]

Combining these results, we get:

$$\frac{t_{tr}}{T_{orb}} = \frac{t_{tr}}{\frac{2\pi}{\sqrt{GM_*}} R_{orb}^{3/2}} = \frac{\frac{R_*}{R_{orb}}}{\pi}$$ [2 points]

And solving for R_{orb}:

$$R_{orb} = \frac{GM_* t_{tr}^2}{2^2 R_*^2}$$ [2 points]

Finally putting this last result back into the expression for the orbital period:

$$T = \frac{\pi G M_*}{4R^3} t_{tr}^3$$ [2 points]

At this point, just a numerical evaluation is needed, though a more elegant solution can be achieved by means of scaling relations by noting that this very same expression must be true in the case of the Earth-Sun system as seen from far away. Having that 31 minutes is the 4% of 12.94h:

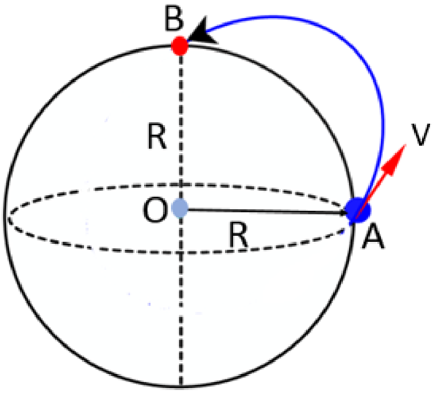
$$T = 365.25 \cdot \frac{0.1}{0.1^3} \cdot 0.04^3 \sim 2.3\ días \sim 2d\ 7h$$ [2 points]

These values were inspired by the planetary system Trappist 1.

SOLUTION

TQ11 [15 points]

11.1



[1 point]

$$EM = -\frac{GMm}{R} + \frac{1}{2}mv^2$$ [1 point]

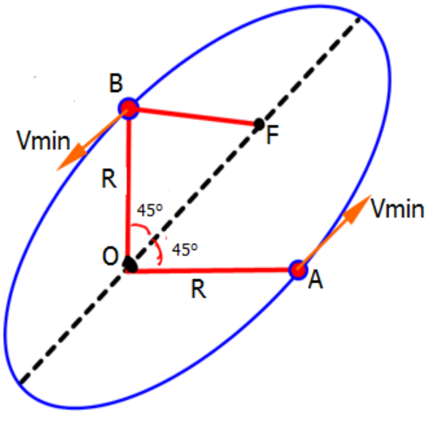
At points A and B, the v is the same; so, if v is minimum, EM is minimum:

We know that the mechanical energy for an elliptical orbit is given by:

$$EM = -\frac{GMm}{2a}$$ [1 point]

where 2a is the length of the major axis.

Since EM must be minimal, then 2a must be minimal [1 point]



[1 point]

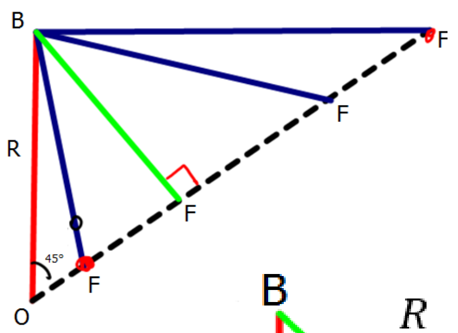
O: Center of the Earth and one of the focus of the ellipse.

F: the other focus of the ellipse

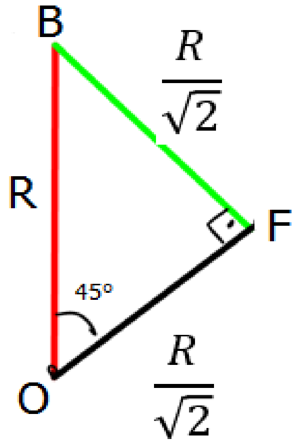
$$OB + BF = 2a$$ [1 point]

$OB + BF = 2a$, must be minimal.

F takes an arbitrary position, so to minimize, BF must be perpendicular to OF:



[2 points]



[1 point]

$$OB + BF = R + \tfrac{R}{\sqrt{2}}$$ [1 point]

$$2a = R + \tfrac{R}{\sqrt{2}}$$

This is the minimum value it can take

Correspondingly, the expression for the minimum velocity is

$$v = \sqrt{\frac{2GM}{(1+\sqrt{2})R}}$$ [1 point]

Using the M and R values for the Earth, the minimum velocity is

$$v = 7195\ \tfrac{m}{s}$$ [1 point]

11.2 Furthermore, $OF = 2\,e{\cdot}a$, where e is the eccentricity

$$\frac{R}{\sqrt{2}} = 2\,e\,\frac{R}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$ [2 points]

$$e = \frac{1}{1+\sqrt{2}} = 0.414$$ [1 point]

SOLUTION

TQ12 [15 points]

12.1 For a satellite that describes a uniform circular motion of radius r around the star and mass M, we have:

$$v = \left[R_{Hodograph}\right] = \sqrt{\frac{GM}{r}}$$ [1 point]

12.2

$$\vec{a} = -\frac{Gm}{r^2}\hat{r}$$ [1 point]

$$L = mr^2\omega$$ [1 point]

$$\rightarrow \vec{a} = -\frac{GMm\omega}{L}\hat{r} = \frac{\Delta\vec{v}}{\Delta t}$$

$$\frac{GMm}{L}\frac{\Delta\theta}{\Delta t} = \left|\frac{\Delta v}{\Delta t}\right|$$

$$\Delta v = \pm\frac{GMm}{L}\Delta\theta$$

$$k = \frac{GMm}{L}$$ [2 points]

12.3

$$L = mv_cR$$ [1 point]

$$k_c = \frac{GMm}{L} = v_c = \sqrt{\frac{GM}{R}}$$ [1 point]

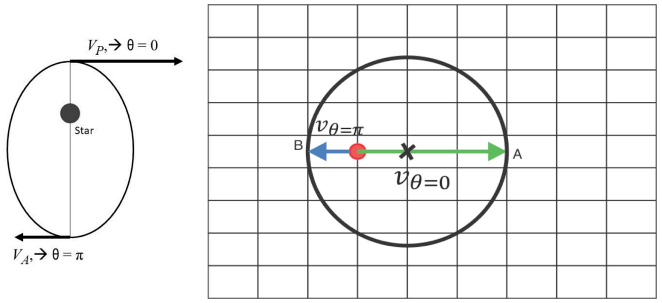
12.4 In the following scheme, the hodographic circumference has been constructed as follows:

- For $\theta = 0$, the velocity vector is drawn in arbitrary units (3, for instance) corresponding to the velocity in the periastron, v_P . Its head determines the point A.
- For $\theta = \pi$, the velocity vector is drawn diametrically opposite in the apastron, v_A .also in arbitrary units. Its head determines point B. The hodograph must pass through points A and B. Therefore, the radius of the hodograph must be equal to:

  $$R_{Hodograph} = \frac{v_P+v_A}{2}$$ [1 point]

  And the "distance" *d* from the star to the center of the circumference will be:

  $$d = \frac{v_P-v_A}{2}$$ [1 point]



[2 points]

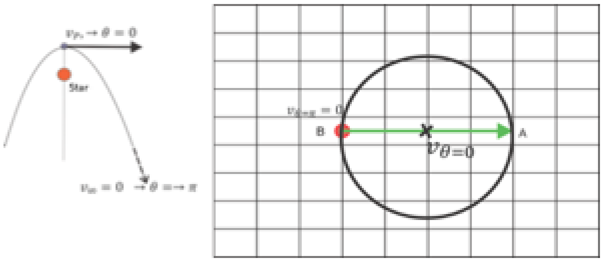
12.5 In the following scheme, the hodographic circumference has been constructed as follows:

- For $\theta = 0$, the velocity vector is drawn in arbitrary units (4, for instance) corresponding to the velocity in the periastron, that is v_P. Its head determines the point A. Then the velocity vector is drawn diametrically opposite in the apastron, that is, $v_A = 0$. Its head determines point B that coincides with the star. The hodograph must pass through points A and B. Therefore, the radius of the hodograph must be equal to:

$$R_{Hodograph} = \frac{v_P}{2}$$ [1 point]

And the distance d from the star to the center of the hodographic circumference

$$d = \frac{v_P}{2}$$ [1 point]



[2 points]

SOLUTION

TQ 13 [15 points]

13.1 A single electron deposits energy in the CCD, as follows:

$$E_{deposited} = \text{stopping power x } \rho_{si} \text{ x thickness}$$

$$E_{deposited} = 3.012 \frac{MeV\,cm^2}{g} \times \frac{2.34g}{cm^3} \times 0.06cm = 422.9\ keV \quad \text{[4 points]}$$

Since an electron with 15 MeV energy deposits 422.9 keV, we must calculate the number of pairs electron/hole

$$422.9keV \times \frac{1}{2.36eV} = 1.79x10^5 \frac{e}{h} \quad \text{[4 points]}$$

How many pixels will be able to excite $1.79x10^5$ pairs of e/h?

$$1.79x10^5 x \frac{1}{250} = 716\ pixels \quad \text{[2 points]}$$

13.2 The number of electrons entering the detector area is:

$$flux \times area \times t \quad \text{[1 point]}$$

$$\frac{600\ e}{cm^2 s} \times [1.3 \times 1.3]\ cm^2 \times 0.03s = 30\ e^- \quad \text{[1 point]}$$

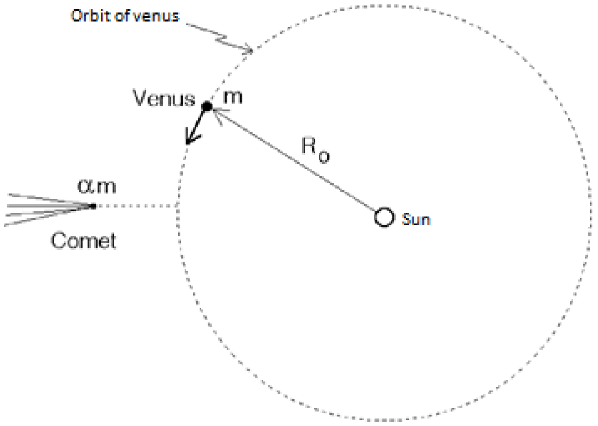
~30 electrons enter the detector area, and we know that one electron excites 716 pixels, the total number of excited pixels is:

$$716 \frac{pixels}{electron} \times 30.42\ electrons = 2.18 \times 10^4 pixels \quad \text{[1 point]}$$

If the CCD has a total of 1024x1024 pixels = $1.048 \times 10^6 pixels$ then the total percentage of pixels that will be driven in a single image is ~ 2.08% [2 points]

SOLUTION

TQ14 [35 points]



14.1 Let M be the mass of the Sun, then, equating the gravitational force with the force Centripetal

$$-\frac{GMm}{R_0^2}\hat{r} = -\frac{mv_0^2}{R_0}\hat{r}$$

$$v_0^2 = \frac{GM_{sun}'}{R_0}$$ [1 point]

14.2 The mechanical energy of Venus (before collision) is:

$$E_i = -\frac{GMm}{R_0^2} + \frac{1}{2}mv_0^2 = -\frac{GMm}{2R_0^2}$$ [1 point]

14.3 Since the comet (when far away) moves radially towards the sun, it has no angular momentum (with respect to the origin in the Sun). Then the angular momentum of Venus is the same as Venus

$$L = R_0 m v_0$$ [1 point]

This allows us to find the component angular of the Venus velocity just after the collision. The angular momentum just after the collision is

$$L = R_0(m + \alpha m)v_\theta$$ [2 points]

Since the angular momentum is conserved it follows that

$$v_\theta = \frac{v_0}{1+\alpha}$$ [2 points]

The conservation of the linear momentum in the radial direction must be realized that the interaction between Venus and the comet are internal forces and, therefore, to calculate the velocity of the comet we can ignore the effect introduced by the interaction between the comet and Venus. The comet has zero energy, so

$$K = - U = + \frac{GM\alpha m}{R_0} = \frac{1}{2}\alpha m v_c^2$$ [2 points]

v is the velocity of the comet just before the collision ignoring the effect introduced by Venus). It follows that

$$v_C^2 = \frac{2GM}{R_0}$$ [1 point]

We now apply the conservation of linear momentum along the radial direction

$$\alpha m v_C = (m + \alpha m)v_r$$ [1 point]

where v_r is the velocity of Venus-2 just after the collision. It follows that

$$v_r = \frac{\alpha}{1+\alpha} v_C$$ [1 point]

14.4 Venus-2 mechanical energy (we evaluate it just after the collision) is:

$$E_f = U + K = - \frac{GMm(1+\alpha)}{R_0} + \frac{1}{2}m(1 + \alpha)\left(v_\theta^2 + v_r^2\right)$$ [2 points]

$$= -(1 + \alpha)\frac{GMm}{R_0} + \frac{1}{2}(1 + \alpha)\left[2\left(\frac{\alpha}{1+\alpha}\right)^2 + \frac{1}{(1+\alpha)^2}\right]\frac{GMm}{R_0}$$ [2 points]

$$= - \frac{GMm}{2R_0}\left(\frac{1+4\alpha}{1+\alpha}\right) = E_i\left(\frac{1+4\alpha}{1+\alpha}\right)$$ [1 point]

14.5 ¨Venus-2¨ orbit is obviously no longer a circle. Since the energy is negative it must, therefore, be elliptical. It has to

$$\frac{E_i}{E_f} = \frac{a_f}{a_i} = \frac{a_f}{R_0}$$ [2 points]

Here a_i and a_f are the semi-major axes of the orbits of Venus and Venus-2, respectively. It follows that

$$a_f = R_0 \frac{E_i}{E_f} = R_0 \frac{1+\alpha}{1+4\alpha}$$ [3 points]

14.6 Using Third law is Kepler, we have

$$\frac{T'}{T} = \left(\frac{a_f}{r_0}\right)^{3/2} = \left(\frac{1+\alpha}{1+4\alpha}\right)^{3/2}$$ [3 points]

the year is shortened.

14.7 For the 'Venus-2' to collide with the Sun, the perihelion radius of post-collision orbit should be

$$r_p = R_\odot = 6.955\times10^8 m$$

$$= \frac{6.955\times10^8}{0.723x1.496\times10^{11}} R_0$$ [1 point]

$$r_p = 0.00643 R_0$$ [0.5 points]

$$r_a = R_0$$ [0.5 points]

Hence

$$a_c = \left(r_p + r_a\right)/2 = 0.5032 R_0$$ [1 point]

$$\frac{a_c}{R_0} = 0.5032 = \frac{1+\alpha_c}{1+4\alpha_c}$$

$$\alpha_c = \frac{1-0.5032}{4\times0.5032-1}$$

$$\alpha_c = 0.4905$$ [2 points]

14.8 For post-collision orbit,

$$v_\theta = \frac{1}{1+\alpha_c} v_0$$

$$v_r = \frac{\sqrt{2\alpha_c}}{1+\alpha_c} v_0$$

$$v_f = \sqrt{v_r^2 + v_\theta^2}$$

$$v_f = \frac{\sqrt{2\alpha_c^2+1}}{1+\alpha_c} v_0$$ [1 point]

Thus,

$$\delta v = v_0 - v_f$$

$$= \left(1 - \frac{\sqrt{2\alpha_c^2+1}}{1+\alpha_c}\right) v_0$$

$$= (1 - 0.8165) v_0 = 0.1835 v_0$$

$$\frac{\delta v}{v_0} = 18.35\%$$ [2 points]

$$\delta\theta = \left(\frac{v_r}{v_\theta}\right)$$

$$= \left(\sqrt{2}\alpha_c\right)$$

$$\delta\theta = 34.74°$$ [2 points]

SOLUTION

TQ15 [55 points]

PART A:

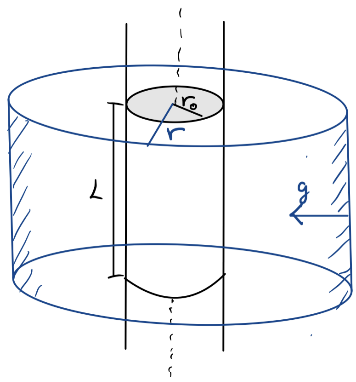
A.1 In cylindrical coordinates, the gravitational field is given by

$$\vec{g} = \vec{g}(r) = g(r)\,\hat{r} = g\,\hat{r}, \qquad \text{[1 point]}$$

where, in view of the axial symmetry, g can be computed from the Gauss law for the gravitational field:

$$\vec{g} \bullet \vec{A} = -\ 4\pi G M_{in}, \qquad \text{[1 point]}$$

where M_{in} is the mass enclosed by the surface A, as shown in the figure below:



Taking the surface to be a cylinder of length L and radius r, from the above equation:

$$2\pi r L g = -\ 4\pi G M_{in}$$

$$g = -\frac{2GM_{in}}{rL} \qquad \text{[1 point]}$$

There are two cases (regions):

- For $r > r_0$, one has $M_{in} = \mu L$ and so [0.5 points]

$$g = -\frac{2G\mu}{r} \qquad \text{[1 point]}$$

- For $r < r_0$, one has $M = \frac{r^2}{r_0^2}\mu L$ and so [0.5 points]

$$g = -\frac{2G\mu}{r_0}\left(\frac{r}{r_0}\right) \qquad \text{[1 point]}$$

A.2 Writing $\vec{g}(r_0)$, in terms of G, μ and r_0:

$$\vec{g}(r_0) = -\frac{2G\mu}{r_0}\hat{r}$$

$$g_0 \equiv \left|\vec{g}(r_0)\right| = \frac{2G\mu}{r_0}$$ [1 point]

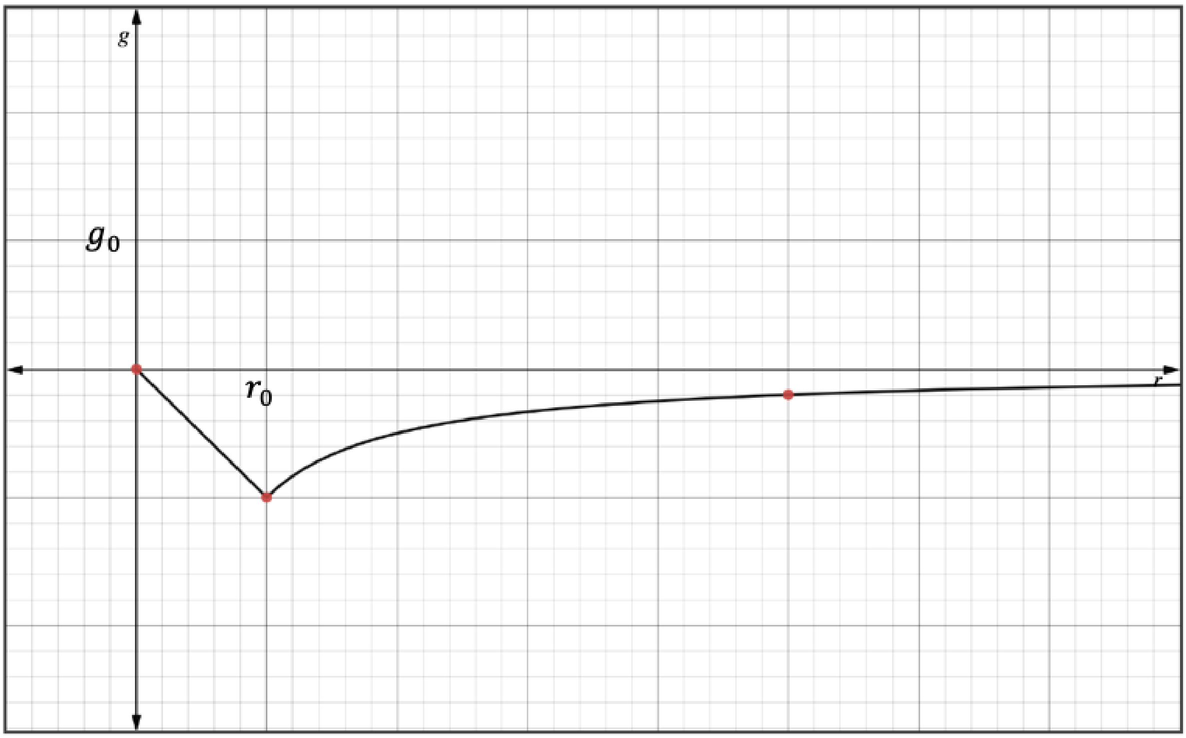
A.3

The following points should be included in the sketch:

- $(0, 0)$ [0.5 points]
- $(1, -1)$ [0.5 points]

Additionally,

- the sketch is linear between $(0, 0)$ and $(1, -1)$ [0.8 points]
- the sketch is concave for $r > r_0$, $\left(\sim \frac{1}{r}\right)$ [0.8 points]
- the sketch is drawn below x-axis. [0.4 points]



A.4 The centripetal acceleration equals the gravitational acceleration, thus

$$\frac{v^2}{R} = \frac{2G\mu}{R}$$ [1 point]

$$\frac{2\pi R}{\tau} = \sqrt{2G\mu}$$ [1 point]

$$R = \frac{\sqrt{2G\mu}}{2\pi}\tau$$

$$A = \frac{\sqrt{2G\mu}}{2\pi}$$ [1 point]

$$\alpha = 1$$ [1 point]

A.5 The potential energy of the particle is given by

$$U(r) = -\int_b^r \vec{F}(r)\cdot d\vec{r}$$

$$= m\int_b^r g(r)dr$$ [1 point]

$$= 2Gm\mu\int_b^r \frac{1}{r}dr$$ [1 point]

$$= 2Gm\mu \ln\left(\frac{r}{b}\right)$$ [1 point]

Where b is a constant that sets the 0 of U.

Note 1: Usually, $U = 0$ is set at $b = \infty$. For a cosmic string, this is not possible. Instead, we can choose $U = 0$, for example, at the string's surface $b = r_0$ *(this choice is not relevant!).* Thus

$$U = 2Gm\mu \ln\left(\frac{r}{r_0}\right)$$

Note 2: For completeness, notice that the potential inside the string, $r < r_0$ is

$$U = 2Gm\mu\left(\frac{r}{r_0}\right)^2 - U_0,$$

where $U_0 = 2Gm\mu\left(\frac{r}{r_0}\right)^2 - 2\,Gm\mu\,\ln\left(\frac{r}{r_0}\right)$ is a constant (again, not relevant!) that can be chosen such that U is continuous at the surface of the string. However, for the solution of the problem, it is not necessary to show this.

A.6 The total energy of the particle is conserved, thus:

$$\frac{1}{2}mv^2 + 2Gm\mu \ \ln\left(\frac{r}{b}\right) = 2Gm\mu \ln\left(\frac{R_{max}}{b}\right) \qquad \text{[2 point]}$$

Solving for R_{max}:

$$R_{max} = Re^{\frac{v^2}{4G\mu}} \qquad \text{[2 point]}$$

Notice that the answer does not depend on b.

A.7 NO. [1 point]

From the previous result,

$$R_{max} = Re^{\frac{v^2}{4G\mu}},$$

we see that it is **not possible** to escape the gravitational field since for any speed, there is always a maximum distance $R_{max} < \infty$

PART B:

B.1 The energy density is given by

$$\rho = aT^4 \qquad \text{[1 point]}$$

Equivalently:

$$\rho = \frac{4\sigma}{c}T^4 = \frac{\pi^2 k_B^4}{15\,\hbar^3 c^3}T^4 \qquad \text{[1 point]}$$

Note 3: This result arrives from the integration of the spectral energy density given by the Planck law over all frequencies. However, it is not required to show this integral.

B.2

$$r_0 = \frac{\hbar^{n_1} c^{n_2}}{k_B T}$$

- r_0 has dimensions of length: $[l]$
- $\hbar$ has dimensions of energy $\times$ time: $[e\ t]$
- c has dimensions of speed: $[l/t]$
- $k_B T$ has dimensions of energy: $[e]$

Then:

$$[l] = \frac{[e\,t]^{n_1}\left[\frac{l}{t}\right]^{n_2}}{[e]}$$

Since the LHS and the RHS should have the same dimensions, we get:

$$[l]: 1 = n_2$$
$$[e]: 0 = n_1 - 1$$
$$[t]: n_1 = n_2 \qquad \text{[2 points]}$$

The solution is

$$n_1 = 1 \qquad \text{[1 point]}$$
$$n_2 = 1 \qquad \text{[1 point]}$$

B.3 The energy of a piece of the string of length L is

$$E = \rho\pi r_0^2 L = Mc^2 \qquad \text{[1 point]}$$

where $M = \mu L$ is the mass of the piece. Solving for μ:

$$\mu = \frac{\rho\pi r_0^2}{c^2} \qquad \text{[1 point]}$$

B.4 The weak field condition is

$$\frac{2G\mu}{c^2} \ll 1$$
$$2G\frac{\rho\pi r_0^2}{c^2} \ll 1 \qquad \text{[1 point]}$$
$$2G\frac{\pi}{c^2}\left(aT^4\right)\left(\frac{\hbar c}{k_B T}\right)^2 \ll 1 \qquad \text{[1 point]}$$
$$\frac{2Ga\hbar^2\pi}{c^2 k_B^2}\,T^2 \ll 1 \qquad \text{[2 point]}$$

Where we used: $\mu = \frac{\rho\pi r_0^2}{c^2}$, $\rho = aT^4$, $r_0 = \frac{\hbar c}{k_B T}$

$$\frac{2Ga\hbar^2\pi}{c^2 k_B^2}\,T^2 \ll 1$$
$$\frac{2G\hbar^2\pi}{c^2 k_B^2}\left(\frac{\pi^2 k_B^4}{15\,\hbar^3 c^3}\right)T^2 \ll 1$$
$$\frac{2\pi^3}{15}\left(\frac{Gk_B^2}{\hbar c^5}\right)T^2 \ll 1$$
$$\frac{2\pi^3}{15}\,\frac{T^2}{T_{Pl}^2} \ll 1 \qquad \text{[2 point]}$$

where $T_{Pl} = 1.416784\times10^{32} K$ (known as the Planck Temperature)

Note 4: the numerical factor $\frac{2\pi^3}{15} \sim 4.13$, *(a 5% error tolerance is accepted)*
Note 5: From **A.4** we get that

$$2G\mu = v^2$$

where v is the speed with which a particle would orbit a string. Thus the weak field is rewritten as

$$\frac{v^2}{c^2} \ll 1$$

which is equivalent to a non-relativistic condition.

B.5 The weak field condition is

$$4.13\left(\frac{T}{T_{Pl}}\right)^2 \ll 1$$

equivalently

$$\sim 2\frac{T}{T_{Pl}} \ll 1$$ [1 point]

i. $\frac{2T_{EW}}{T_{Pl}} \sim 2\times\frac{10^{15}}{10^{32}} \approx 1.4\times10^{-17} \ll 1$ [1 point]

ii. $\frac{2T_{GUT}}{T_{Pl}} \sim 2\times\frac{10^{29}}{10^{32}} \approx 1.4\times10^{-3} \ll 1$ [1 point]

Note 6: Using the following expression $4.13\left(\frac{T}{T_{Pl}}\right)^2 \ll 1$

One gets

i. $4.13\left(\frac{T_{EW}}{T_{Pl}}\right)^2 \sim 2.1\times10^{-34} \ll 1$

ii. $4.13\left(\frac{T_{GUT}}{T_{Pl}}\right)^2 \sim 2.1\times10^{-6} \ll 1$

And are also acceptable answers *(a 5% error tolerance is accepted)*

B.6

i. Yes [0.5 point]

ii. Yes [0.5 point]

PART C:

C.1 From the figure, the asymptotic condition for the observer to see a second image is that the light ray travelling from O directly to S should bend and travel along SE. This is the maximum angle of bending.

As all angles are small, we can safely use

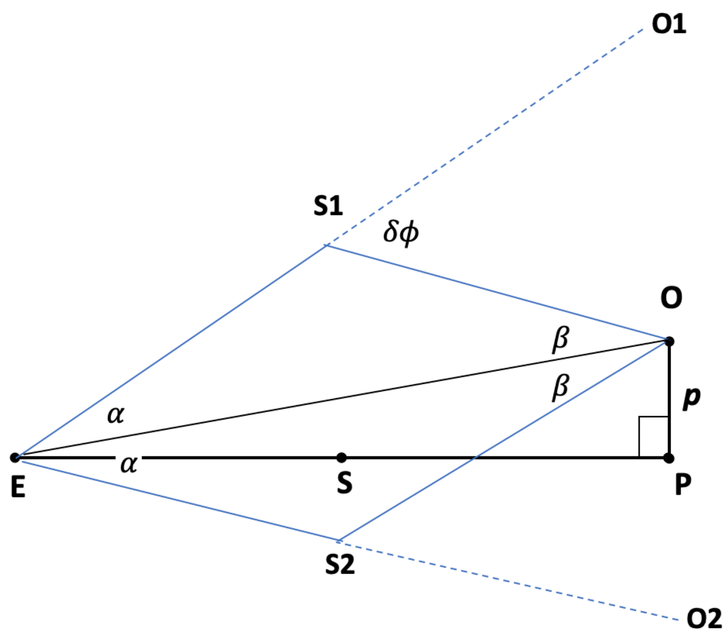
$$\sin x \approx \tan x \approx x$$

Now,

$$\frac{p}{SP} < \delta\phi \qquad \text{[2 points]}$$
$$< \frac{4\pi G\mu}{c^2}$$
$$< 2\pi\left(\frac{2G\mu}{c^2}\right)$$
$$< 2\pi\left(\frac{2\pi^3}{15}\right)\left(\frac{T^2}{T_{Pl}^2}\right) \qquad \text{[2 points]}$$

And $SP \approx \left(D_{OE} - D_{ES}\right)$

$$p < \frac{4\pi^4}{15T_{Pl}^2}T^2\left(D_{OE} - D_{ES}\right) \qquad \text{[2 points]}$$



C.2 In the figure, the blue lines represent the bending of two light rays corresponding to the images O1 and O2. Note that $D_{ES} \approx D_{ES1} \approx D_{ES2}$ and $D_{OS} \approx D_{OS1} \approx D_{OS2}$. Thus the angle $S1EO \approx S2EO \equiv \alpha$ and $S1OE \approx S2OE \equiv \beta$.2α is the angular separation we are looking for.

[2 points]

Further, notice

$$\alpha + \beta = \delta\phi$$ [1 point]

Using law of sines in the triangle **EOS1**

$$\frac{\alpha}{D_{S1O}} = \frac{\beta}{D_{ES1}}$$ [1 point]

we get

$$2\alpha = 2\delta\phi\left(\frac{D_{OE}-D_{ES}}{D_{OE}}\right)$$ [2 points]

C.3 If $D_{OE} = 2D_{ES}$,

$$2\alpha = \delta\phi = 2\pi\left(\frac{2\pi^3}{15}\right)\left(\frac{T^2_{GUT}}{T^2_{Pl}}\right)\approx 1.29\times10^{-5}$$ [2 points]

and

$$\delta\phi = 1.22\frac{\lambda}{D}$$

Thus, using $\lambda\in[3\times10^{-7}m, 8\times10^{-7}m]$

$$D = 1.22\frac{\lambda}{\delta\phi} \in \left[3.75\times10^{-2}m, 7.51\times10^{-2}m\right]$$ [2 points]

Any answer within this interval is valid.

A remarkable result of this model is light deflection by a cosmic string, which leads to the possibility of detection through gravitational lensing. For instance, cosmic string trings moving across the line of sight with respect to an Earth-based observer will cause line-like discontinuities as shown in the figure below (taken from [1]).

Gravitational lensing by cosmic strings 1737

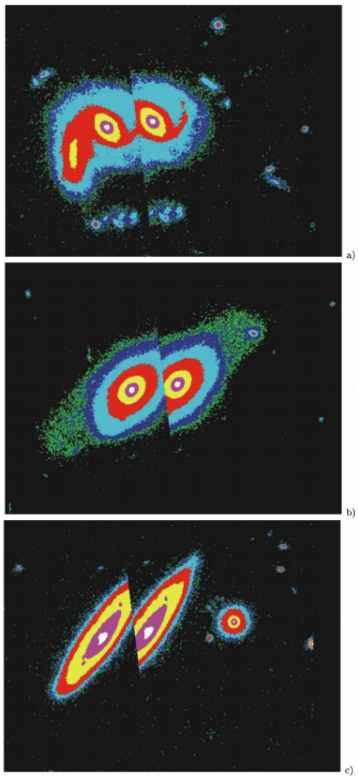


Figure 6. (a) This picture represents the lensed image of a spiral galaxy. The direction of a string is almost perpendicular to galaxy plane. Duplicated details are clearly visible. (b) In this case, the string is inclined with respect to the galaxy plane. As a result sharp edge appears. (c) Also in this case the direction of string is inclined with respect to galactic plane.

[1] M. V. Sazhin, O. S. Khovanskaya, M. Capaccioli, G. Longo, M. Paolillo, G. Covone, N. A. Grogin, E. J. Schreier, Gravitational lensing by cosmic strings: what we learn from the CSL-1 case, *Monthly Notices of the Royal Astronomical Society*, Volume 376, Issue 4, March 2007, Pages 1731–1739, https://doi.org/10.1111/j.1365-2966.2007.11543.x